Decoupling of unphysical states in the minimal pure spinor formalism II

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ABSTRACT: This is the second of a series of two papers where decoupling of unphysical states in the minimal pure spinor formalism is investigated. The multi-loop amplitude prescription for the minimal pure spinor superstring formulated in hep-th/0406055 involves the insertion of picture changing operators in the path integral. In the first paper it was shown that these operators are not BRST closed inside correlators. Therefore a new proof of decoupling of BRST exact states is needed. In this paper we present such a proof, which applies to arbitrary genus. It relies in part on a (previously unnoticed) invariance of the path integral measure.

Contents

1 Introduction

A new superstring formalism, the pure spinor formalism, has been developed over the past ten years [1–3], see [4, 5] for reviews. In this new formalism, the theory exhibits manifest super Poincaré invariance, as in the Green-Schwarz (GS) formalism, but in contrast with the GS string the worldsheet theory in flat target space is free, as in the Ramond-Neveu-Schwarz (RNS) formalism, so the theory can be quantized straightforwardly.

There are two versions of this formalism, the minimal [2] and the non-minimal formalism [3]. In this paper we discuss exclusively the minimal case. A multi-loop scattering amplitude prescription was developed in [2] and it involves introducing a number of picture changing operators (PCO's) that are inserted in the path integral. These operators are BRST closed in a distributional sense and depend on constant spinors (C_α) and constant tensors (B_{mn}). It was argued in [2] that amplitudes are independent of C and B, because the Lorentz variation of PCO's is BRST exact. The same conclusion can be reached from the results in [6], where the presence of the PCO's in the path integral was derived via BRST-BV quantization that takes into account the gauge invariances due to zero modes [7]. A manifestly Lorentz invariant prescription can be obtained by integrating over C and B [2] and this is the form we will use here.

One may question whether the PCO's Y are closed because their BRST variation only vanishes in a distributional sense, $QY \sim x\delta(x)$, where x is an appropriate field variable. That would give zero only if integrated against a smooth function of x. It is shown in [8] however that the amplitudes in the current minimal amplitude prescription do contain singular terms that imply that QY is not zero inside correlators. This leads to explicit

dependence of the amplitudes on B and C and problems with decoupling of BRST exact states and with Lorentz invariance, when one does not integrate over B and C. The singular behavior of the amplitudes is linked to the fact that the gauge fixing condition of the zero mode gauge invariances implicit in the current formulation is singular and we anticipate that a proper treatment of global issues will allow for a non-singular gauge fixing condition and will lead to a prescription free of these problems. Work in this direction is in progress. Here we will show that these problems are also absent when one integrates over B and C.

Note that $QY \neq 0$ by itself does not imply that Q exact states do not decouple. It only implies that the standard argument for decoupling of unphysical states that involves integrating Q by parts does not automatically lead to decoupling. In [8] it was shown that at tree level one can nevertheless establish decoupling of BRST exact states using integration of Q by parts. The core of that argument is the vanishing of the trace of a certain Lorentz invariant tensor, (ϵT), defined in (2.13). At one loop, we can identify a Lorentz invariant tensor (ϵTR), defined in (2.20), that is the one-loop analog of (ϵT). That is to say, all one-loop amplitudes with a Q exact state are proportional to the trace of this tensor. Had this trace vanished, this would imply that Q exact states decoupled at one loop. It turns out however, as proven in [8], that the trace of this one-loop invariant tensor does not vanish, so one needs a different argument to prove decoupling of unphysical states.

The main result of this paper is to provide such an argument. The new argument does not use integration of Q by parts. Rather it makes use of a (so far unnoticed) invariance of the path integral measure and the fact the zero mode integrals act as projectors on a certain Lorentz scalar. Then one can show that the integrand that results from BRST exact insertions does not contain this scalar, hence amplitudes that contain unphysical states vanish after integration.

This paper is organized as follows. In the next section we present a short review of the amplitude prescription in the minimal pure spinor formalism, mostly to set our notation. In section 3 the main result is discussed, namely the proof of decoupling of unphysical states in the minimal pure spinor formalism. Finally there is a short concluding section.

2 Minimal pure spinor amplitude prescription

In this section we review the amplitude prescription for the minimal pure spinor superstring [2]. The N point tree-level amplitude is given by

$$\mathcal{A} = \langle V_1(z_1)V_2(z_2)V_3(z_3) \int dz_4 U_4(z_4) \cdots \int dz_N U_N(z_N) Y_{C_1}(y_1) \cdots Y_{C_{11}}(y_{11}) \rangle \qquad (2.1)$$

The vertex operators are given by

$$V = \lambda^\alpha A_\alpha(x,\theta), \qquad (2.2)$$

$$U = \partial\theta^\alpha A_\alpha(x,\theta) + \Pi^m A_m(x,\theta) + d_\alpha W^\alpha(x,\theta) + \frac{1}{2} N^{mn} \mathcal{F}_{mn}(x,\theta). \qquad (2.3)$$

where $A_\alpha(x,\theta), A_m(x,\theta)$ is the superfield connection for $N=1, d=10$ SYM and $W^\beta, \mathcal{F}_{mn}$ are the corresponding field strengths. Y_C are the picture changing operators (PCO) needed

to absorb the zero modes of the weight zero fields:

$$Y_C(y) = C_\alpha \theta^\alpha(y) \delta(C_\beta \lambda^\beta(y)), \tag{2.4}$$

where C_α is a constant spinor.

The N point one-loop amplitude involves one unintegrated vertex operator, V, $N-1$ integrated vertex operators, U, a composite $\tilde{b}$ field and picture changing operators Y_C, Z_B and Z_J,

$$\mathcal{A}^{(N)} = \int d^2\tau \langle | \int d^2 u \mu(u) \tilde{b}_{B^1}(u, z_1) \prod_{P=2}^{10} Z_{B^P}(z_P) Z_J(z_{11}) \prod_{I=1}^{11} Y_{C_I}(y) |^2$$
$$V_1(t_1) \prod_{T=2}^{N} \int d^2 t_T U_T(t_T) \rangle. \tag{2.5}$$

The Z_B's and Z_J are PCO's needed to absorb the zero modes of the weight one fields:

$$Z_B(z) = \frac{1}{2} B_{mn} \lambda(z) \gamma^{mn} d(z) \delta(B_{mn} N^{mn}(z)), \quad Z_J(z) = \lambda^\alpha(z) d_\alpha(z) \delta(J(z)), \tag{2.6}$$

where B_{mn} is a constant antisymmetric tensor. The composite $\tilde{b}$ field satisfies

$$\{Q, \tilde{b}_B(u, z)\} = T(u) Z_B(z). \tag{2.7}$$

This equation ensures the Q variation of the b ghost vanishes after integrating over moduli space. The solution is given by [2]

$$\tilde{b}_B(u, z) = b_B(u) + T(u) \int_u^z dv B_{pq} \partial N^{pq}(v) \delta(BN(v)). \tag{2.8}$$

The local b ghost, $b_B(u)$, is a composite operator, constructed out of the worldsheet fields:

$$b_B(z) = b_{B0}(z)\delta(BN(z)) + b_{B1}(z)\delta'(BN(z)) + b_{B2}(z)\delta''(BN(z)) + b_{B3}(z)\delta'''(BN(z)), \tag{2.9}$$

where the primes denote derivatives, $BN \equiv B_{mn} N^{mn}$ and the explicit expressions of b_{Bi} can be found in appendix A.

The amplitude prescription for $g > 1$ is given by

$$\mathcal{A}^{(N)} = \int d^2\tau_1 \cdots d^2\tau_{3g-3} \langle | \prod_{P=1}^{3g-3} \int d^2 u_P \mu_P(u_P) \tilde{b}_{B^P}(u_P, z_P)$$
$$\prod_{P=3g-2}^{10g} Z_{B^P}(z_P) \prod_{R=1}^{g} Z_{J^R}(v_R) \prod_{I=1}^{11} Y_{C_I}(y) |^2 \prod_{T=1}^{N} \int d^2 t_T U_T(t_T) \rangle. \tag{2.10}$$

where we have grouped the insertions in a way that will be useful later.

The amplitude prescription just described was also obtained from first principles in [6] by coupling the pure spinor sigma model to topological gravity. In particular, the PCO's arise in this context by gauge fixing gauge invariances due to zero modes. The constant tensors C_α and B_{mn} enter the theory through gauge fixing conditions. Thus, provided

there are no BRST anomalies and the gauge fixing is non-singular, the amplitudes should be independent of C and B. However, as shown in [8], the gauge fixing is singular and amplitudes do depend on the choice of B and C. One can restore manisfest Lorentz invariance by integrating over all possible choices for B and C. This integral is incorporated in the computations below.

As described in [2], the amplitude (2.5) is evaluated by first using the OPE's to remove all fields of non-zero weight. After this step all fields have weight zero. This can be evaluated by replacing the fields by their zero modes and performing the zero mode integrations. So we need to know how to integrate over the zero modes. For the d, θ, x variables this is standard, so we only discuss the integration over λ, N, B, C.

A typical integral one encounters is given by [2]:

$$\mathcal{A} = \int [d\lambda][dB][dC] \prod_{R=1}^{g} [dN_R] f(\lambda, N_R, J_R, C, B) \tag{2.11}$$

where the zero mode measure for $[d\lambda]$ is given by

$$[d\lambda]\lambda^{\alpha}\lambda^{\beta}\lambda^{\gamma} = d\lambda^{\alpha_1} \wedge \cdots \wedge d\lambda^{\alpha_{11}} (\epsilon T)^{\alpha\beta\gamma}_{\alpha_1 \cdots \alpha_{11}}, \tag{2.12}$$

with

$$(\epsilon T)^{\alpha\beta\gamma}_{\alpha_1 \cdots \alpha_{11}} = \epsilon_{\alpha_1 \cdots \alpha_{16}} T^{\alpha\beta\gamma\alpha_{12} \cdots \alpha_{16}}, \tag{2.13}$$

$$T^{\alpha\beta\gamma\alpha_{12} \cdots \alpha_{16}} = \gamma_m^{\alpha[\alpha_{12}} \gamma_n^{|\beta|\alpha_{13}} \gamma_p^{|\gamma|\alpha_{14}} (\gamma^{mnp})^{\alpha_{15}\alpha_{16}]}. \tag{2.14}$$

The zero mode measure for (each zero mode of) $[dN]$ is given by

$$[dN]\lambda^{\alpha_1} \cdots \lambda^{\alpha_8} = dN^{m_1 n_1} \wedge \cdots \wedge N^{m_{10} n_{10}} \wedge dJ R^{\alpha_1 \cdots \alpha_8}_{m_1 n_1 \cdots m_{10} n_{10}}, \tag{2.15}$$

with

$$R^{\alpha_1 \cdots \alpha_8}_{m_1 n_1 \cdots m_{10} n_{10}} \equiv \gamma^{((\alpha_1 \alpha_2}_{m_1 n_1 m_2 m_3 m_4} \gamma^{\alpha_3 \alpha_4}_{m_5 n_5 n_2 m_6 m_7} \gamma^{\alpha_5 \alpha_6}_{m_8 n_8 n_3 n_6 m_9} \gamma^{\alpha_7 \alpha_8))}_{m_{10} n_{10} n_4 n_7 n_9} + \text{permutations}. \tag{2.16}$$

The permutations make R antisymmetric under exchange in both $m_i \leftrightarrow n_i$ and $m_i n_i \leftrightarrow m_j n_j$ and the double brackets denote subtraction of the gamma trace. The zero mode integral (2.11) is only non-zero if the function f depends on (λ, N, J, C, B) as

$$f(\lambda, N, J, C, B) = h(\lambda, N, J, C, B) \prod_{R=1}^{g} \partial^{M_R} \delta(J) \prod_{P=1}^{10} \prod_{R=1}^{g} \partial^{L_{P,R}} \delta(B^P N_R) \prod_{I=1}^{11} \partial^{K_I} \delta(C^I \lambda), \tag{2.17}$$

where the polynomial h assumes the form

$$(\lambda)^{8g-8+\sum_{I=1}^{11}(K_I+1)} \prod_{R=1}^{g} (J_R)^{M_R} (N_R)^{\sum_{P=1}^{10} L_{P,R}} \prod_{P=1}^{10} (B^P)^{L_{P,R+1}} \prod_{I=1}^{11} (C^I)^{K_I+1}. \tag{2.18}$$

The integration over the zero modes of the pure spinor variables and the constant tensors is defined in [2] as

$$\mathcal{A} = c \frac{\partial}{\partial \lambda^{\alpha_1}} \cdots \frac{\partial}{\partial \lambda^{\alpha_3}} (\epsilon T)^{\alpha_1 \cdots \alpha_3}_{\beta_1 \cdots \beta_{11}} \left[R^{\alpha_4 \cdots \alpha_{11}}_{m_1 n_1 \cdots m_{10} n_{10}} \frac{\partial}{\partial \lambda^{\alpha_4}} \cdots \frac{\partial}{\partial \lambda^{\alpha_{11}}} \frac{\partial}{\partial B^1_{m_1 n_1}} \cdots \frac{\partial}{\partial B^{10}_{m_{10} n_{10}}} \right]^g$$

$$\frac{\partial}{\partial C^1_{\beta_1}} \cdots \frac{\partial}{\partial C^{11}_{\beta_{11}}} \prod_{I=1}^{11} \left(\frac{\partial}{\partial \lambda^\delta} \frac{\partial}{\partial C^I_\delta} \right)^{K_I} \prod_{P=1}^{10}$$

$$\prod_{R=1}^{g} \left(\frac{\partial}{\partial B^P_{pq}} \frac{\partial}{\partial N^{pq}_R} \right)^{L_{P,R}} \prod_{R=1}^{g} \left(\frac{\partial}{\partial J_R} \right)^{M_R} h(\lambda, N_R, J_R, C, B), \qquad (2.19)$$

for some proportionality constant c.

In the sequel we will use the following notation

$$(\epsilon T R)^{\alpha_1 \cdots \alpha_{11}}_{\beta_1 \cdots \beta_{11} m_1 n_1 \cdots m_{10} n_{10}} \equiv (\epsilon T)^{((\alpha_1 \alpha_2 \alpha_3}_{\beta_1 \cdots \beta_{11}} R^{\alpha_4 \cdots \alpha_{11}))}_{m_1 n_1 \cdots m_{10} n_{10}}, \qquad (2.20)$$

$$(T R)^{\alpha_1 \cdots \alpha_{11} \beta_{12} \cdots \beta_{16}}_{m_1 n_1 \cdots m_{10} n_{10}} \equiv T^{\beta_{12} \cdots \beta_{16} ((\alpha_1 \alpha_2 \alpha_3} R^{\alpha_4 \cdots \alpha_{11}))}_{m_1 n_1 \cdots m_{10} n_{10}}. \qquad (2.21)$$

3 Proof of decoupling of Q exact states

Decoupling of unphysical states in the minimal pure spinor formalism would follow, if all insertions were Q closed. As discussed in [8], however, the Y's are not BRST closed inside correlators. More specifically the PCO's for the λ zero modes, denoted by Y_C, are not closed:

$$Q Y_C = Q(C\theta)\delta(C\lambda) = (C\lambda)\delta(C\lambda). \qquad (3.1)$$

This vanishes in a distributional sense but it does not necessarily vanish if there are insertions containing factors of $\frac{1}{C\lambda}$. In the minimal pure spinor formalism the pure spinor zero mode measure $[d\lambda]$ contains a factor $\frac{1}{(\lambda^+)^3}$. Therefore one cannot conclude $Q Y_C = 0$. In fact in [8] some amplitudes with Q exact states were computed in the formulation without an integral over C. These did not vanish, hence we can conclude $Q(Y_{C^1} \cdots Y_{C^{11}}) \neq 0$. Nevertheless, we were able to show decoupling of Q exact states at tree level in the formulation with an integral over C. In next subsection we review the tree-level argument in a form that generalizes to the higher loops and show that Q exact states decouple to all orders. A crucial role in this proof is played by symmetry of the insertions that follows from the particular form of the picture raising operators, Z_B. We will first present the proof for one-loop amplitudes, followed by a proof of decoupling valid at any genus.

3.1 Tree-level amplitudes

After integrating out the non-zero modes every tree-level amplitude assumes the form

$$\mathcal{A} = \int [d\lambda][dC] d^{16}\theta \lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha\beta\gamma}(\theta, a, k) \theta^{\beta_1} \cdots \theta^{\beta_{11}} C^1_{\beta_1} \cdots C^{11}_{\beta_{11}} \delta(C^1 \lambda) \cdots \delta(C^{11}\lambda), \quad (3.2)$$

where a denotes all polarizations and k denotes all momenta. Note that we have assumed that integration over the non-zero modes does not affect the factor of $Y_{C^1} \cdots Y_{C^{11}}$. This

can be justified either by writing Y_C as a function of only zero modes or by inserting the factor of $(Y_C)^{11}$ at $z = \infty$ on the worldsheet. The three factors of λ originate from the three unintegrated vertex operators and the factors of θ, C and $\delta(C\lambda)$ from the eleven picture changing operators Y_C. In order to evaluate (3.2) first note that only terms with five θ's can contribute:

$$\mathcal{A} = \int [d\lambda][dC] d^{16}\theta \lambda^\alpha \lambda^\beta \lambda^\gamma f^{(5)}_{\alpha\beta\gamma\beta_{12}\cdots\beta_{16}}(a,k) \theta^{\beta_1} \cdots \theta^{\beta_{16}} C^1_{\beta_1} \cdots C_{\beta_{11}} \delta(C^1\lambda) \cdots \delta(C^{11}\lambda),$$
(3.3)

We will show now that the integration is a projection on the scalar in $f^{(5)}_{\alpha\beta\gamma\beta_{12}\cdots\beta_{16}}(a,k)$. To this end we write the tensor product $(\lambda)^3(\theta)^5$ in terms of its irreducible representations:

$$\lambda^\alpha \lambda^\beta \lambda^\gamma \theta^{\beta_{12}} \cdots \theta^{\beta_{16}} = T^{\alpha\beta\gamma\beta_{12}\cdots\beta_{16}} T_{\alpha'\beta'\gamma'\beta'_{12}\cdots\beta'_{16}} \lambda^{\alpha'} \lambda^{\beta'} \lambda^{\gamma'} \theta^{\beta'_{12}} \cdots \theta^{\beta'_{16}} + \tag{3.4}$$
$$(T_1)^{\alpha\beta\gamma\beta_{12}\cdots\beta_{16}[mn]} (T_1)_{\alpha'\beta'\gamma'\beta'_{12}\cdots\beta'_{16}[mn]} \lambda^{\alpha'} \lambda^{\beta'} \lambda^{\gamma'} \theta^{\beta'_{12}} \cdots \theta^{\beta'_{16}} +$$
$$\sum_{i\geq 2} (T_i)^{\alpha\beta\gamma\beta_{12}\cdots\beta_{16}x_i} (T_i)_{\alpha'\beta'\gamma'\beta'_{12}\cdots\beta'_{16}x_i} \lambda^{\alpha'} \lambda^{\beta'} \lambda^{\gamma'} \theta^{\beta'_{12}} \cdots \theta^{\beta'_{16}},$$

where x_i are the indices representing the representation. To obtain the above expansion one first needs to compute the tensor product $\text{Gam}^3\mathbf{16} \otimes \text{Asym}^5\mathbf{16}$. This contains one scalar, hence there is only one invariant tensor with the indices and symmetries of T (an explicit realization is specified in (2.14)). One also finds there is one $\mathbf{45}$ in the tensor product, hence the second line. The sum in the last line runs over all the other irreps in the tensor product, each one has an invariant tensor (T_i) associated to it. Furthermore all the (T_i)'s satisfy

$$T^{\alpha\beta\gamma\beta_{12}\cdots\beta_{16}} (T_i)_{\alpha\beta\gamma\beta_{12}\cdots\beta_{16}x_i} = 0. \tag{3.5}$$

This can be proven by contracting both sides of (3.4) with $T_{\alpha\beta\gamma\beta_{12}\cdots\beta_{16}}$. The integrations in (3.3) can be evaluated by Lorentz invariance:

$$\left(\int d^{16}\theta \theta^{\beta_1} \cdots \theta^{\beta_{16}} \right) \left(\int [d\lambda][dC] \lambda^\alpha \lambda^\beta \lambda^\gamma C^1_{\beta_1} \cdots C_{\beta_{11}} \delta(C^1\lambda) \cdots \delta(C^{11}\lambda) \right) = \tag{3.6}$$
$$\epsilon^{\beta_1\cdots\beta_{16}} (\epsilon T)^{\alpha\beta\gamma}_{\beta_1\cdots\beta_{11}} = T^{\alpha\beta\gamma\beta_{12}\cdots\beta_{16}}$$

After using (3.5) one sees all the non scalar terms in (3.4) are annihilated by the integration. It is therefore a projection on the scalar as claimed. The final expression for the amplitude becomes

$$\mathcal{A} = T^{\alpha\beta\gamma\beta_{12}\cdots\beta_{16}} f^{(5)}_{\alpha\beta\gamma\beta_{12}\cdots\beta_{16}}(a,k). \tag{3.7}$$

3.1.1 Decoupling of Q exact states at tree level

After integrating out the non-zero modes, the amplitude containing a Q-exact state becomes,

$$\int [d\lambda] d^{16}\theta (Q\Omega(\lambda,\theta,a,k)) \theta^{\beta_1} \cdots \theta^{\beta_{11}} C^1_{\beta_1} \cdots C_{\beta_{11}} \delta(C^1\lambda) \cdots \delta(C^{11}\lambda), \tag{3.8}$$

for some Ω, where all fields are zero modes. Our task now is to show that this integral vanishes for any Ω.

Since only the terms with five θ's and three λ's in $Q\Omega$ contribute, we focus on terms in Ω with two λ's and six θ's. The upshot of the proof is that no Lorentz scalar can be constructed from two λ's and six θ's. Therefore there will be no scalar in $Q(\lambda)^2(\theta)^6$ and since the integration projects on the scalar the amplitude vanishes. In order to make this argument precise let us write:

$$\Omega|_{(\lambda)^2(\theta)^6} = \lambda^\alpha \lambda^\beta \theta^{\beta_1} \cdots \theta^{\beta_6} \tilde{f}_{\alpha\beta\beta_1\cdots\beta_6}(a, k) \tag{3.9}$$

for some $\tilde{f}$. The next step is writing the tensor product $(\lambda)^2(\theta)^6$ in terms of its irreducible representations:

$$\Omega|_{(\lambda)^2(\theta)^6} = \tilde{f}_{\alpha\beta\beta_1\cdots\beta_6}(a, k) \left(\sum_i (\tilde{T}_i)^{\alpha\beta\beta_1\cdots\beta_6 y_i} (\tilde{T}_i)_{\alpha'\beta'\beta'_1\cdots\beta'_6 y_i} \lambda^{\alpha'} \lambda^{\beta'} \theta^{\beta'_1} \cdots \theta^{\beta'_6} \right). \tag{3.10}$$

In the above formula it is important to note that there are no scalars in the tensor product of two pure spinors and six fermionic spinors. This is reflected by the fact that y_i represents (a positive number of) indices for every i. We now perform the Q transformation:

$$Q\Omega|_{(\lambda)^2(\theta)^6} = \tilde{f}_{\alpha\beta\beta_1\cdots\beta_6}(a, k) \left(\sum_i (\tilde{T}_i)^{\alpha\beta\beta_1\cdots\beta_6 y_i} (\tilde{T}_i)_{\alpha'\beta'[\gamma'\beta'_2\cdots\beta'_6] y_i} \lambda^{\alpha'} \lambda^{\beta'} \lambda^{\gamma'} \theta^{\beta'_2} \cdots \theta^{\beta'_6} \right). \tag{3.11}$$

After invoking (3.7) we find

$$\int [d\lambda] d^{16}\theta \left(Q\Omega|_{(\lambda)^2(\theta)^6} \right) \theta^{\beta_1} \cdots \theta^{\beta_{11}} C^1_{\beta_1} \cdots C_{\beta_{11}} \delta(C^1\lambda) \cdots \delta(C^{11}\lambda) = \tag{3.12}$$

$$\tilde{f}_{\alpha\beta\beta_1\cdots\beta_6}(a, k) \sum_i (\tilde{T}_i)^{\alpha\beta\beta_1\cdots\beta_6 y_i} (\tilde{T}_i)_{\alpha'\beta'[\gamma'\beta'_2\cdots\beta'_6] y_i} T^{\alpha'\beta'[\gamma'\beta'_2\cdots\beta'_6]} = 0$$

This vanishes because

$$T^{\alpha'\beta'[\gamma'\beta'_2\cdots\beta'_6]} = 0, \tag{3.13}$$

which follows from the statement that there are no scalars in $(\lambda)^2(\theta)^6$. This concludes the proof that (3.8) vanishes.

3.2 Higher-loop amplitudes

In order to prove decouling of unphysical states at higher-loop amplitudes one can take similar steps to the tree-level case. This means that one first reduces the amplitude to a zero mode integral, which is effectively a projection onto a scalar and then one shows there is no scalar when one started with a Q exact state. In the higher-loop case we need an additional ingredient for the second step which is a symmetry possessed by the integrand of the functional integral.

Additional symmetry. The amplitude prescription contains products of PCO's Z_B and Z_J. The main observation is that

$$Z_B Z_J = B_{mn} \lambda \gamma^{mn} d \; \delta(B_{mn} N^{mn})(\lambda d)\delta(J) \tag{3.14}$$

is invariant under

$$\delta B_{mn} = (\lambda\gamma_{[m})_\alpha f_{n]}^\alpha. \tag{3.15}$$

where $f^{n\alpha}$ are constants. This transformation acts on the $B_{mn}N^{mn}$ and $B_{mn}\lambda\gamma^{mn}d$ as,

$$\delta B_{mn}N^{mn} = (\lambda\gamma_m)_\alpha f_n^\alpha(\lambda\gamma^{mn}w) = (\lambda\gamma^n f_n)(\lambda w), \tag{3.16}$$

$$\delta B_{mn}(\lambda\gamma^{mn}d) = (\lambda\gamma_m)_\alpha f_n^\alpha(\lambda\gamma^{mn}d) = (\lambda\gamma^n f_n)(\lambda d). \tag{3.17}$$

Since all these transformation contain either (λw) or (λd) and Z_J contains both $\delta(\lambda w)$ and λd:

$$\delta(Z_B Z_J) = 0. \tag{3.18}$$

Now recall that at genus g, $3g-3$ B's (one at genus one) enter via the b ghost. We take these B's to be inert. The remaining $7g+3$ B's (9 at genus one) are taken to transform as in (3.15) . Note that at one loop, the factor of $(Z_B)^9 Z_J$ is placed at a single point on the worldsheet. At two-loop order, the additional factor of $(Z_B)^7 Z_J$ is placed at a second point on the worldsheet. And at each additional loop order, one places the new factor of $(Z_B)^7 Z_J$ at a gth point on the worldsheet. With this choice, (3.15) is an invariance of the theory for $7g+3$ B's and the amplitudes must respect this symmetry.

One can understand the origin of this symmetry by going back to the first principles derivation of the amplitude prescription in [6]. As shown there, PCO insertions arise from gauge fixing the invariance due to pure spinor zero modes. In particular, this leads to the insertion in the path integral of the expression

$$\exp\left(\pi_\alpha\lambda^\alpha + \tilde{\pi}_\alpha\theta^\alpha + \sum_{I=1}^g\left(-\pi_{mn}^I\frac{1}{2}d^I\gamma^{mn}\lambda + \tilde{\pi}_{mn}^I N^{mnI} - \pi^I d_\alpha^I\lambda^\alpha + \tilde{\pi}^I J^I\right)\right), \tag{3.19}$$

where $\pi_\alpha, \tilde{\pi}_\alpha, \pi^{mnI}, \tilde{\pi}^{mnI}, \pi^I, \tilde{\pi}^I$ are the BRST auxiliary fields, g is the genus and I counts the g zero modes of worldsheet vectors. This expression is invariant under

$$\delta\pi_{mn}^I = (\lambda\gamma_{[m})_\alpha f_{n]}^{I\alpha}, \qquad\qquad \delta\pi^I = -\frac{1}{2}(\lambda\gamma^n f_n^I) \tag{3.20}$$

$$\delta\tilde{\pi}_{mn}^I = (\lambda\gamma_{[m})_\alpha \tilde{f}_{n]}^{I\alpha}, \qquad\qquad \delta\tilde{\pi}^I = -(\lambda\gamma^n \tilde{f}_n^I) \tag{3.21}$$

This symmetry implies that out of the 45 components of each π_{mn}^I and $\tilde{\pi}_{mn}^I$ only 10 are independent, as it should be since the number of BRST auxiliary fields should be equal to the number of gauge fixing conditions.

In the non-minimal formalism, π_{mn}^I and $\tilde{\pi}_{mn}^I$ are identified with $\bar{N}^{mnI}$ and S_{mn}^I and these automatically have the correct number of independent components because they are constructed using the non-minimal pure spinor variables. On the other hand, to arrive at the minimal formulation we set

$$\pi_{mn}^I = p^{jI}B_{mn}^{jI}, \qquad \tilde{\pi}_{mn}^I = \tilde{p}^{jI}B_{mn}^{jI}, \qquad j = 1,\ldots,10 \tag{3.22}$$

and integrate over $p^{jI}, \tilde{p}^{jI}$. This leads to the Z_B insertions. In this case, there is a remnant of the symmetry (3.20), which is (3.15) for all B. This suggests that the amplitudes is also invariant, if we transform the $(3g-3)$ (one when $g=1$) factors of B involved in the b insertions[1], but we will not prove or use this here.

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[1] Recall that $(3g-3)$ (one when $g=1$) of the Z_B factors are absorbed into the b-insertions.

3.2.1 One-loop amplitudes

After integrating out all non-zero modes, as well as the d_α zero modes, every one-loop amplitude can be written as

$$
\int [d\lambda][dN][dC][dB]d^{16}\theta \lambda^{\alpha_1}\cdots\lambda^{\alpha_{11}} B^1_{m_1 n_1}\cdots B^{10}_{m_{10}n_{10}} f^{m_1 n_1 \cdots m_{10}n_{10}}_{\alpha_1\cdots\alpha_{11}}(\theta,a,k)
$$
$$
\theta^{\beta_1}\cdots\theta^{\beta_{11}} C^1_{\beta_1}\cdots C^{11}_{\beta_{11}}\delta(C^1\lambda)\cdots\delta(C^{11}\lambda)\delta(B^1 N)\cdots\delta(B^{10}N)\delta(J), \quad (3.23)
$$

where all fields are zero modes and the integrand is invariant under the B transformation (3.15). As in the tree amplitude, we are assuming that integration over the non-zero modes does not affect the $(Y_C)^{11}$ factor since this factor can be written in terms of only zero modes. In this expression, eleven factors of λ originate as follows: one from the unintegrated vertex operator, one from Z_J and nine from the nine factors of Z_B. In general the zero mode integral can contain additional factors of the Lorentz currents N, higher powers of B and higher derivatives of $\delta(BN)$. These additional factors can be put into the form of (3.23) by integrating by parts using that $N^{pq}B_{mn}\partial\delta(BN) = -\delta^{[p}_m\delta^{q]}_n\delta(BN)$.

One can show that the integral in (3.23) is also a projection on a scalar. To see this first note that there is one scalar in $\mathrm{Gam}^{11}\mathbf{16}\otimes\mathrm{Asym}^5\mathbf{16}\otimes\mathrm{Asym}^{10}\mathbf{45}$. This implies one can write

$$
\lambda^{\alpha_1}\cdots\lambda^{\alpha_{11}}\theta^{\beta_{12}}\cdots\theta^{\beta_{16}} B^1_{m_1 n_1}\cdots B^{10}_{m_{10}n_{10}} = (TR)^{\alpha_1\cdots\alpha_{11}\beta_{12}\cdots\beta_{16}}_{m_1 n_1 \cdots m_{10}n_{10}}\left((TR)(\lambda)^{11}(\theta)^5(B)^{10}\right) +
$$
$$
\sum_i (S_i)^{\alpha_1\cdots\alpha_{11}\beta_{12}\cdots\beta_{16}}_{m_1 n_1\cdots m_{10}n_{10}x_i}\left(S_i(\lambda)^{11}(\theta)^5(B)^{10}\right)^{x_i},
$$
$$
(3.24)
$$

where the notation $\left((TR)(\lambda)^{11}(\theta)^5(B)^{10}\right)$ means that all indices of (TR) have been contracted with those of λ, θ and B and $\left(S_i(\lambda)^{11}(\theta)^5(B)^{10}\right)^{x_i}$ denotes an object that has x_i as its only free index and which transforms in some non-scalar representation. Similar to the tree-level case the invariant tensors S_i satisfy

$$
((RT)(S_i))^{x_i} = 0. \qquad (3.25)
$$

Note that since B is not a covariant tensor this is not the decomposition of a Lorentz invariant object into a lot of Lorentz invariant terms like (3.4). However this does not matter, the point of performing this expansion is that all the non scalar terms vanish due to the integration. The last point follows from (3.25) and

$$
\int [d\lambda][dC][dB][dN]\lambda^{\alpha_1}\cdots\lambda^{\alpha_{11}} B^1_{m_1 n_1}\cdots B^{10}_{m_{10}n_{10}} C^1_{\beta_1}\cdots C^{11}_{\beta_{11}} \qquad (3.26)
$$
$$
\delta(C^1\lambda)\cdots\delta(C^{11}\lambda)\delta(B^1 N)\cdots\delta(B^{10}N)\delta(J) = (\epsilon TR)^{\alpha_1\cdots\alpha_{11}}_{\beta_1\cdots\beta_{11}m_1 n_1\cdots m_{10}n_{10}},
$$

which is also a consequence of the fact there is only one Lorentz scalar in $\mathrm{Gam}^{11}\mathbf{16}\otimes\mathrm{Asym}^5\mathbf{16}\otimes\mathrm{Asym}^{10}\mathbf{45}$.

Decoupling of Q exact states. We will show that if

$$\lambda^{\alpha_1} \cdots \lambda^{\alpha_{11}} B^1_{m_1 n_1} \cdots B^{10}_{m_{10} n_{10}} f^{m_1 n_1 \cdots m_{10} n_{10}}_{\alpha_1 \cdots \alpha_{11}} (\theta, a, k) \qquad (3.27)$$

can be written as $Q\Omega$ where Ω is invariant under the B transformation then (3.23) vanishes.

Note Ω must contain ten λ's, six θ's and ten B's. There are two scalars in $\mathrm{Gam}^{10}\mathbf{16} \otimes \mathrm{Asym}^6\mathbf{16} \otimes \mathrm{Asym}^{10}\mathbf{45}$. Since $\mathrm{Gam}^{11}\mathbf{16} \otimes \mathrm{Asym}^5\mathbf{16} \otimes \mathrm{Asym}^{10}\mathbf{45}$ contains only a single scalar and Q maps scalars to scalars, there is a basis of invariant tensors such that one of the scalars is annihilated by the BRST operator and the other one, call it Ω_1, has non-zero BRST variation, $Q\Omega_1 \neq 0$. This scalar is[2]

$$\Omega_1 = \left(T(\lambda)^3(\theta)^5 \right) \left(R(B)^{10}(\lambda)^7(\theta)^1 \right). \qquad (3.28)$$

Here $\left(R(B)^{10}(\lambda)^7(\theta)^1 \right)$ denotes the unique scalar obtained by contracting all indices of the objects involved. The state $Q\Omega_1$ is a candidate BRST exact state that may not decouple. The scalar Ω_1 however is not invariant under the transformation (3.15) for 9 of the 10 B's. In fact, one can show that Ω_1 is invariant under the transformation (3.15) for only 6 of the 10 B's. To see this, note that $\left(R(B)^{10}(\lambda)^7(\theta)^1 \right)$ can be expressed as

$$(\lambda\gamma^{m_1 \cdots m_5}\lambda)(\lambda\gamma^{m_6 \cdots m_{10}}\lambda)(\lambda\gamma^{m_{11} \cdots m_{15}}\lambda)(\lambda\gamma^{m_{16} \cdots m_{20}}\theta) \qquad (3.29)$$

contracted with the 20 vector indices of $(B)^{10}$. If both indices of B_{pq} are contracted with $m_1 \cdots m_{15}$, then Ω_1 is invariant under the transformation (3.15) for that B since $(\lambda\gamma^{mn_1 \cdots n_4}\lambda)(\lambda\gamma_m)_\alpha = 0$. However, if at least one index of B_{pq} is contracted with $m_{16} \ldots m_{20}$, then Ω_1 is not invariant under the transformation (3.15) for that B. Using the definition of $R^{\alpha_1 \cdots \alpha_8}_{m_1 \ldots m_{20}}$, one finds there are four B's whose indices are contracted with $m_{16} \ldots m_{20}$, so Ω_1 is invariant under the transformation (3.15) for 6 of the 10 B's.

But since the gauge parameter must be invariant under (3.15) for 9 of the 10 B's, there is no way to generate Ω_1 as a possible gauge parameter. We thus conclude that if it is Q exact and invariant under the B transformation,

$$f^{m_1 n_1 \cdots m_{10} n_{10}}_{\alpha_1 \cdots \alpha_{11}}(\theta, a, k)\lambda^{\alpha_1} \cdots \lambda^{\alpha_{11}} B^1_{m_1 n_1} \cdots B^{10}_{m_{10} n_{10}} \qquad (3.30)$$

does not contain any scalars constructed from eleven λ's, five θ's and ten B's. Since the integration projects on the (single) scalar the total zero mode integral vanishes. The precise argument is analogous to the steps in section 3.1.1.

3.2.2 Higher-loop amplitudes

The argument for $g > 1$ is exactly analogous. After integrating out all non-zero modes, as well as the zero modes of d_α, every $g > 1$ loop amplitude can be written as

$$\int d^{16}\theta [d\lambda][dC]\lambda^{\alpha_1}\lambda^{\alpha_2}\lambda^{\alpha_3}\theta^{\beta_1} \cdots \theta^{\beta_{11}} C^1_{\beta_1} \cdots C^{11}_{\beta_{11}} \delta(C^1\lambda) \cdots \delta(C^{11}\lambda)$$

$$\prod_{I=1}^{g} \left([dB^I][dN^I]\lambda^{\alpha_4^I} \cdots \lambda^{\alpha_{11}^I} B^{1I}_{m_1^I n_1^I} \cdots B^{10I}_{m_{10}^I n_{10}^I} \delta(B^{1I}N) \cdots \delta(B^{10I}N)\delta(J^I) \right)$$

$$f^{m_1^1 n_1^1 \cdots m_{10}^g n_{10}^g}_{\alpha_1 \alpha_1 \alpha_3 \alpha_4^1 \cdots \alpha_8^g}(\theta, a, k) \qquad (3.31)$$

---

[2]Another possible candidate, $\left(T(\lambda)^2(\theta)^6 \right) \left(R(B)^{10}(\lambda)^8 \right)$, vanishes identically because of (3.13).

where all fields are zero modes and the integrand is invariant under the B transformation (3.15). Now the factors λ originate from the $(7g+3)$ factors of Z_B and the g factors of Z_J. Additional factors of N, B and derivatives of $\delta(BN)$ can be removed as in the one-loop case.

In this case the analogue of (3.26) is

$$
\int [d\lambda][dC]\lambda^{\alpha_1}\lambda^{\alpha_2}\lambda^{\alpha_3}C^1_{\beta_1}\cdots C^{11}_{\beta_{11}}\delta(C^1\lambda)\cdots\delta(C^{11}\lambda),
$$
$$
\prod_{I=1}^{g}\left([dB^I][dN^I]\lambda^{\alpha_4^I}\cdots\lambda^{\alpha_{11}^I}B^{1I}_{m_1^I n_1^I}\cdots B^{10I}_{m_{10}^I n_{10}^I}\delta(B^{1I}N)\cdots\delta(B^{10I}N)\delta(J^I)\right)
$$
$$
= (\epsilon T R^g)^{\alpha_1\alpha_2\alpha_3\alpha_4^1\cdots\alpha_{11}^g}_{\beta_1\cdots\beta_{11}m_1^1 n_1^1\cdots m_{10}^g n_{10}^g} \tag{3.32}
$$

where $(\epsilon T R^g)$ is the generalization of (2.20) involving g factors of R.

There are g candidate BRST exact states that may not decouple, which are the analogs of (3.28) and are given by

$$
\Omega_J = \left(T(\lambda)^3(\theta)^5\right)\prod_{I=1}^{J-1}\left(R(B^I)^{10}(\lambda)^8\right)\ \ \left(R(B^J)^{10}(\lambda)^7(\theta)^1\right)\ \ \prod_{I=J+1}^{g}\left(R(B^I)^{10}(\lambda)^8\right) \tag{3.33}
$$

where B^I denotes the B's associated with the Ith zero mode. As in the one-loop case, the term $\left(R(B^J)^{10}(\lambda)^7(\theta)^1\right)$ is at most invariant under 6 of the 10 B^J transformations. But invariance under (3.15) requires invariance under 7 of the 10 B^J transformations.

So we conclude that unphysical states decouple to all orders in g.

4 Conclusion

We presented in this paper a proof of decoupling of unphysical states in the minimal pure spinor formalism to all loop order. We were able to prove this despite the fact that not all insertions in the path integral are Q closed. More specifically our argument did not involve integrating Q by parts. The two main ingredients were the presence of the B symmetry and the fact that the zero mode integrals act as projectors on a scalar.

As is discussed in [8], the amplitudes in the prescription of [2] without an integral over C are actually singular and the distributional relations do not hold inside correlators. The singularities in the amplitudes are likely to reflect the fact that the gauge choice for the gauge invariances due to zero modes implicit in the prescription of [2] is singular. Obtaining a prescription corresponding to a non-singular gauge choice may require incorporating global issues, in particular taking into account all patches in the pure spinor space[3], see section 6 of [8] for further discussion, and such a prescription is currently under investigation. We anticipate that such a prescription will also lead to decoupling of BRST exact states, without integrating over C and B.

---

[3]We thank Nikita Nekrasov for this suggestion.

Acknowledgments

A   Chain of operators for b ghost

$$
\begin{aligned}
b_{B0} &= \frac{1}{2}G\gamma^{mn}dB_{mn} - \frac{1}{2}H^{\alpha\beta}(\gamma^p\gamma^{mn})_{\alpha\beta}\Pi_p B_{mn} \qquad\qquad\qquad (A.1) \\
&\quad + \frac{1}{2}K^{\alpha\beta\gamma}(\gamma^p\gamma^{mn})_{\beta\gamma}(\gamma_p\partial\theta)_\alpha B_{mn} + \frac{1}{2}S^{\alpha\beta\gamma}(\gamma^p\gamma^{mn})_{\beta\gamma}(\gamma^p\partial\lambda)_\alpha B_{mn}, \\
b_{B1} &= \frac{1}{4}H^{\alpha\beta}(Bd)_\alpha(Bd)_\beta \qquad\qquad\qquad\qquad\qquad\qquad\qquad (A.2) \\
&\quad + \frac{1}{4}K^{\alpha\beta\gamma}(\gamma^p\gamma^{mn})_{\beta\gamma}(Bd)_\alpha\Pi_p B_{mn} + \frac{1}{4}K^{\alpha\beta\gamma}(\gamma^p\gamma^{mn})_{\alpha[\beta}(Bd)_{\gamma]}\Pi_p B_{mn} \\
&\quad + \frac{1}{4}L^{\alpha\beta\gamma\delta}[((\gamma^p\gamma^{mn})_{\gamma\delta}(Bd)_{[\alpha}(\gamma_p\partial\theta)_{\beta]} - (\gamma^p\gamma^{mn})_{\beta[\gamma}(Bd)_{\delta]}(\gamma_p\partial\theta)_\alpha)B_{mn} \\
&\quad - ((\gamma^s\gamma^{rq})_{\alpha[\beta}(\gamma^p\gamma^{mn})_{\gamma]\delta} + (\gamma^s\gamma^{rq})_{\alpha\delta}(\gamma^p\gamma^{mn})_{\beta\gamma})\Pi_p B_{mn}\Pi_s B_{qr}], \\
b_{B2} &= -\frac{1}{8}K^{\alpha\beta\gamma}(Bd)_\alpha(Bd)_\beta(Bd)_\gamma - \frac{1}{8}L^{\alpha\beta\gamma\delta}((\gamma^p\gamma^{mn})_{\gamma\delta}(Bd)_\beta(Bd)_\alpha \qquad (A.3) \\
&\quad + (\gamma^p\gamma^{mn})_{\beta[\gamma}(Bd)_{\delta]}(Bd)_\alpha + \frac{1}{2}(\gamma^p\gamma^{mn})_{\alpha[\delta}(Bd)_\gamma(Bd)_{\beta]})\Pi_p B_{mn}, \\
b_{B3} &= -\frac{1}{16}L^{\alpha\beta\gamma\delta}(Bd)_\alpha(Bd)_\beta(Bd)_\gamma(Bd)_\delta, \qquad\qquad\qquad\qquad (A.4)
\end{aligned}
$$

where $(Bd)_\alpha \equiv B_{mn}(\gamma^{mn}d)_\alpha$. The explicit form of the tensors $G^\alpha, H^{\alpha\beta}, K^{\alpha\beta\gamma}, L^{\alpha\beta\gamma\delta}$ can be found, for example, in section 3 of [9]. They do not contain any B tensors.

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